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ANALYSIS OF BIFURCATIONS FOR LOGISTIC FAMILIES

MD. Asraful Islam & Payer Ahmed

Research Scholar, Department of Mathematics, Jagannath University, Dhaka, Bangladesh

ABSTRACT

Bifurcation means a division, a rending apart. How and when physical, chemical and biological systems sustain sudden barters of the behavior is the appearance of bifurcation. In the case of logistic bifurcation, we are considering the limits or end behaviors of logistic systems. To understand bifurcation behavior, it is often helpful to look at the bifurcation diagram. Saddle-node bifurcation and period-doubling bifurcation are speculated. Bifurcation diagram on logistic map for several iterations are analyzed. We have conferred aggregate espials for different parameter values. Finally, we have exhibited that the schemes are chaotic and non-chaotic for detached parameter appraises.

KEYWORDS: *Bifurcation, Period Doubling Bifurcation, Saddle-Node Bifurcation*

Article History

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INTRODUCTION

A casual characteristic change of a way to the parameter values behavior is called bifurcation. [1] Bifurcations ensue in both incessant systems and detached systems. The name "bifurcation" was first indicated by Henri Poincare in 1885 in the first paper in mathematics showing such a behavior.[2]Henri Poincaré also later named manifold complexion of stagnant points and categorized them. Bifurcation theory has been used to append quantum scheme the dynamics of their transcendent analogues in atomic scheme,[3][4][5] molecular systems,[6] and resonant tunneling diodes.[7] Bifurcation theory has also been applied to the study of laser dynamics[8] and a number of theoretical examples which are difficult to access experimentally such as the kicked top[9] and coupled quantum wells.[10] The prevalent contention for the link between quantum scheme and bifurcations in the prevalent equations of pace is that at bifurcations, the subscription of prevalent orbits becomes large, as Martin Gutzwiller points out in his classic[11] work on quantum chaos.[12]For some pointed out parameter values of r bifurcation arises in a one-parameter family of logistic function. Saddle-node is the just exigent bifurcation. In the saddle-node incident, the way in which the bifurcation arises may be capsize, Also, hoops may endure a period-doubling bifurcation. [13][14]

Definition of saddle-node bifurcation and Period doubling bifurcation.

Saddle-Node Bifurcation

A saddle-node is a conflict and invisibility of two positions in dynamical systems. In disembodied dynamical systems the tant amount bifurcation is continually substitute called a convolution bifurcation. Another name is blue skies bifurcation in allusion to the snappy designing of two fixed points. A saddle-node bifurcation also befalls if the trace of the bifurcation is tumbled. Revolvedacme may endure a saddle-node bifurcation. If the cycle respite is one-dimensional, one of the equivalence mites hectic (the saddle), while the other is durable (the node).Saddle-node bifurcations may be affiliated with hysteresis noose and plague. Bifurcations narrate alteration in the durability or fixity of fixed points as a seizure parameter in the scheme innovates. As an excellently naïve affirmation of a bifurcation in a dynamical system, suspect thing co here not on turret of a vertical beam. The mass of the thing can be envisaged of as the seizure parameter. As the mass of the thing increases, the beam's drift from vertical, which is x, the dynamic variable, resides comparatively consolidated. The mass proximity a certain point – the bifurcation point – the beam will abruptly ouch. Switches in the control parameter in the end switched the characteristic dealing of the system.

Consider the equation of the form $\frac{dx}{dt} = -x^2 + r$

Where r is the seizure parameter. As r is greater than 0, the scheme has one adjusted fixed point and one unadjusted fixed point. As r abates the fixed points shakes simultaneously, shortly clash into a semi-adjusted fixed point at $r = 0$, and to exist when $r < 1$.

Figure 1 Bifurcation Diagram for a Saddle Node Bifurcation of the Equation $\frac{dx}{dt} = -x^2 + r$

Figure 1 shows the behavior of the scheme changes meaningful when the seizure parameter r is 0, 0 is a bifurcation point. This example bifurcation is called the saddle-node bifurcation.

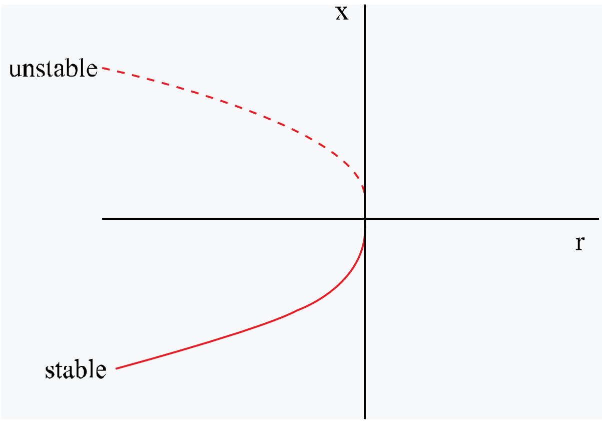

Figure 1

Period Doubling Bifurcation

A period doubling bifurcation in a discrete dynamical system is a bifurcation in which a slight change in a parameter value in the system's equations leads to the system switching to a new behavior with twice the period of the original system. With the doubled period, it bears twice as many repetitions of a process as prior for the numerical values visited by the scheme to revolve them. A period doubling cataract is a succession of doublings and into the bargain doublings of the repeating feast, as the parameter is constant therewithal and therewithal. Period doubling bifurcations can also ensue in incessant dynamical systems, namely as new is tether arises from an existing tether cycle, and the epoch of the newish tether cycle is twice that of the old one. As the parameter innovates, a fixed point may innovate from attracting to repelling and, at the same, give birth to an attracting 2-cyle. Otherwise, the fixed point may innovative from repelling to attracting and, at the equivalent time, give birth to a repelling hoop of period 2.

RESULTS AND DISCUSSIONS

Logistic Family of Functions and Their Bifurcations

Logistic Family of functions is given by $F_r = rx(1-x)$. Here we observe the situations for different parameter values of r . In fact, we will observe attracting and repelling fixed points together with phase portrait. Of course, bifurcation diagram to be sketched in each situation.

Observation 1: the values of r does F_r have an attracting fixed point at $x=0$.

Solution 1: First of all, note that $x=0$ is indeed fixed since $F_r(0)=0$. using the fact that $F'_r(x)=rx(1-x)$. we see that $F'_r(x)=r$, and so 0 is attracting for $-1<r<1$.

Observation 2: The values of r does F_r have a nonzero attracting fixed point.

Solution 2: Let's find the other fixed point of F_r:

$$rx(1-x)=x$$

$$\Rightarrow rx-rx^2-x=0$$

$$\Rightarrow (r-1)x-rx^2=0 \qquad 3.1$$

$$\Rightarrow rx^2+(1-r)x=0$$

$$\Rightarrow (rx+(1-r))x=0$$

$$\Rightarrow x=\frac{r-1}{r} \text{ or } x=0.$$

We may conclude from this that fix $F_r=\left\{0,\frac{r-1}{r}\right\}$. As a check, let's compute

$$F_r\left(\frac{r-1}{r}\right)=r\left(\frac{r-1}{r}\right)\left(1-\frac{r-1}{r}\right)=(r-1)\left(\frac{r-(r-1)}{r}\right)=\left(\frac{r-1}{r}\right).$$

$\frac{r-1}{r}$ Is attracting for $-1<2-r<1 \Rightarrow -3<r<-1 \Rightarrow 1<r<3.$

Observation 3: The bifurcation that occurs when $r=1$.

Solution 3: For $r\neq 0, F_r$ has two fixed points, namely 0 and $\frac{r-1}{r}$. For $1<r<3, \frac{r-1}{r}$ is attracting; likewise, for $-1<r<1, 0$ is attracting. As r decreases through the bifurcation point at $r=1$. The fixed point $\frac{r-1}{r}$ transfers its attractiveness to 0 which continue to attract orbits unit $r\neq -1$. For $r<-1$, both fixed points are repelling.

Observation 4: Phase portrait and bifurcation diagram near at $x = 1$.

Solution 4: See Figure 2 The horizontal asymptote at $x = 1$implies that there is no member of the logistic family having 1 as a fixed point. The vertical asymptote at the origin implies that all members of the logistic family have two fixed points (and the two are unique for all $r \neq -1$ except the degenerate F_0 which is identically zero.

Observation 5: The bifurcation that occurs when $r = 3$.

Recall from observation 3 that 0 is attracting for $-1 < r < 1$, while $\frac{r-1}{r}$ is attracting for $1 < r < 3$. Thus, both fixed points are repelling for $r > 3$.

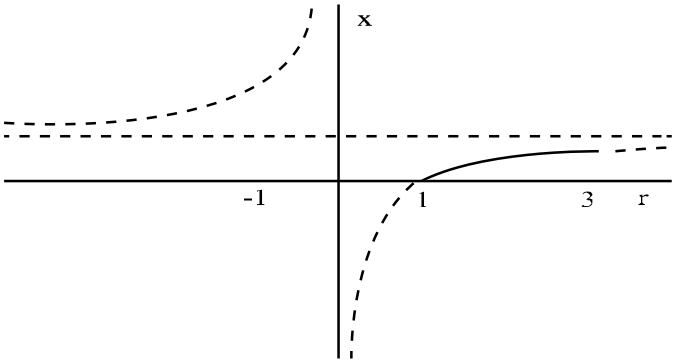

Figure 2: Bifurcation Diagram for the Logistic Family.

Note that

$$F'_r\left(\frac{r-1}{r}\right)\Bigg|_{r=1} = -1$$

Which suggests a 2-cycle may be lurking in the shadows.

Observations 6: An explicit formula for the periodic points of period 2 for F_r.

First compute the second iterate of F_r:

$$F_r^2(x) = r(rx(1-x))(1-rx(1-x)) = r(rx-rx^2)(1-rx+rx^2)$$
$$r(rx - r^2x^2 + r^2x^3 - rx^2 + r^2x^3 - r^2x^4) = r(rx-(r+r^2)x^2 + r^2x^3 - r^2x^4)$$

$$= r^2x - r^2(1+r)x^2 + 2r^3x^3 - r^3x^4.$$

To find the fixed points of F_r^2 (i.e. the period 2 points of F_r), set $F_r^2(x)$ equal to x, rearrange terms and get $(r^2-1)x - r^2(1+r)x^2 + 2r^3x^3 - r^3x^4$. (3.2)This 4th –degree polynomial may be factored into a pair of quadratics. One of these factors must be $(r-1)x - rx^2$ since the fixed points of F_r are also fixed points of F_r^2 (see Equation 3.1). Long dividing out this 2nd-degree polynomial from (3.2), we find that

$$\frac{(r^2-1)x-r^2(1+r)x^2+2r^3x^3-r^3x^4}{(r-1)x-rx^2}=r^2x^2-r(r+1)x+(r+1)$$

And so $[(r-1)x-rx^2][r^2x^2-r(r+1)x+(r+1)]=0.$

This new factor may be solved using the quadratic formula: $x=\frac{r(r+1)\pm\sqrt{r^2(r+1)^2-4r^2(r+1)}}{2r^2}=$

$$=\frac{r(r+1)\pm\sqrt{(r+1)^2 4(r+1)}}{2r^2}=\frac{(r+1)\pm\sqrt{(r+1)(r-3)}}{2r}.$$

Hence, this 2-cycle exists when $r>3$ or $r<-1$ which agree with the results of observation 3.

Proposition: Let $F_r(x)=rx(1-x)$ be a logistic family of functions. Then we have the following observations:

- The values of r does F_r have an attracting fixed point at $x=0$.
- The values of r does F_r have a nonzero attracting fixed point.
- The bifurcation that occurs when $r=1$.
- Phase portrait and bifurcation diagram near at $x=1$.
- The bifurcation that occurs when $r=3$.
- An explicit formula for the periodic points of period two for F_r.

Bifurcation Diagram on Logistic Map for Several Iterations

To understand bifurcation behavior, it is often helpful to look at the bifurcation diagram.

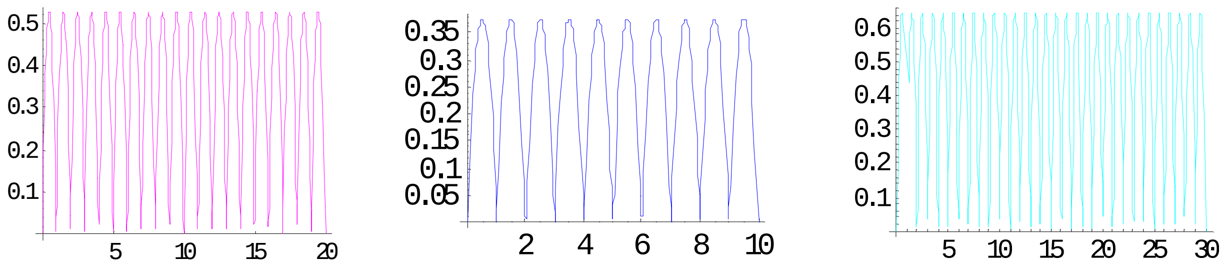

Figure 3: (A) When r = 1.5,10th Iteration, (B) When r = 2.1,20th Iteration, (C) When r = 2.58,30th Iteration.

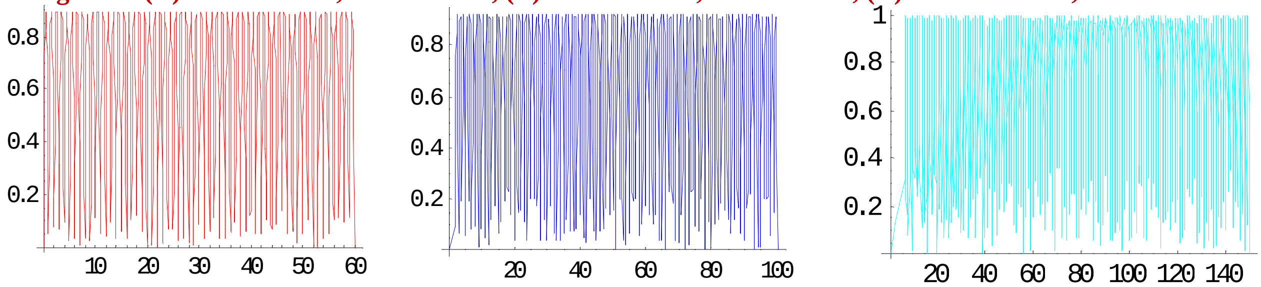

Figure 3: (D) When r = 3.57,60th Iteration, (E) When r = 3.68,100th Iteration, (F) When r = 4,150th Iteration.

Figure 5: A bifurcation in the logistic family $f_r(x) = rx(1-x)$, for several parameter values of r and several iterations.

We have accomplished that when $r = 1.5$ to 4 and $r = 3.57$ to 4 and the number of iterations taken 10 to 50 times, the illustrated figures behooving narrow to narrower.

Bifurcation Diagram of Logistic Function for Different Parameter Values

A period-doubling bifurcation is $f_r(x) = rx(1-x)$ for $r = 3.$ This function is called Logistic function. The Logistic function $f_r(x) = rx(1-x),$ where r is a parameter. For fixed point $f_r(x) = x \Rightarrow x = 0 \;\; or,\; x = \frac{r-1}{r} = P_r$ (say).

If $r > 1,$ then f_r have two fixed points: one at $x = 0$ another at $x = P_r$.

If $r = 1,$ then $x = 0$ is the only fixed point for f_r.

- If $r = 1$, a trans critical bifurcation occurs at $P_1 = 0.$
- If $r = 3$, a period-doubling bifurcation occurs at $P_1 = \frac{2}{3}.$

If $r = 3.45,$ another period-doubling bifurcation occurs and the period two attracting orbit splits into a period four attracting orbit and a period two repelling orbit.

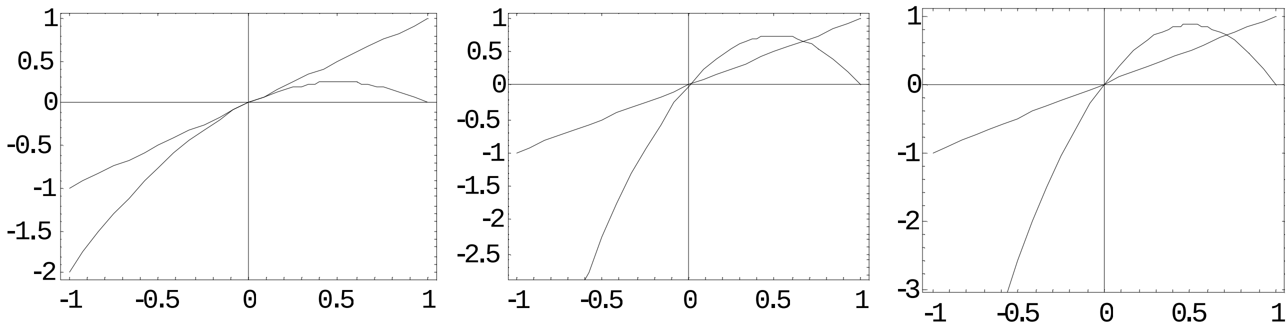

Figure 4: (A) r = 1, (B) r = 3, (C) r = 3.45.

Some Bifurcation Diagrams are plotted for A Few Hundred Iterations.

Figure 5 shows we have observed that bifurcations occur on the interval at $0 \le r \le 4$, $1 \le r \le 4$, $2.5 \le r \le 4$, $2.90 \le r \le 4$, $3 \le r \le 4$, $3.45 \le r \le 4$, $3.54 \le r \le 4$, $3.56 \le r \le 4$, $3.63 \le r \le 4$, $3.74 \le r \le 4$, $3.82 \le r \le 4$, $3.72 \le r \le 3.92$, $3.83 \le r \le 3.96$, $3.80 \le r \le 3.85$, $3.90 \le r \le 4$ and $3.86 \le r \le 3.98$. From the aloft sketch, tidy bifurcations are perceptible in the chaotic virtues of the diagram. Sometimes bifurcation is considered as a fractal. The same is true for all other non-chaotic points. We noticed that the windows of period three at $r = 3.83$, period five at $r = 3.74$, and period six at $r = 3.63$ in the above diagram. Sooth to say, chaotic nature ensues between the interim 3.57 and 4.

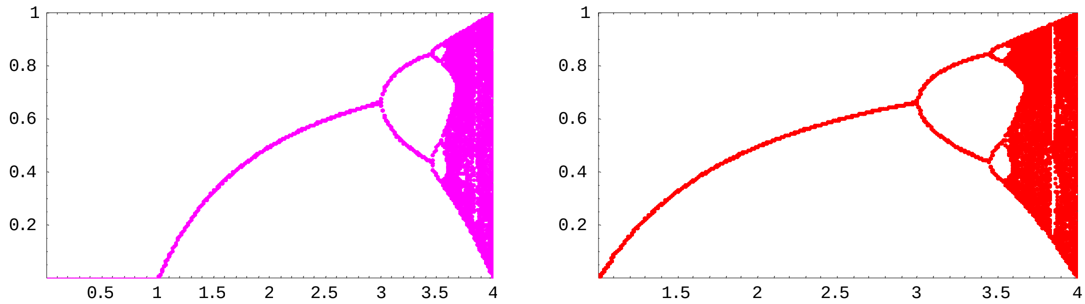

Figure 5: Bifurcation Diagram of Logistic Families for Different Parameter Values.
(A) Logistic Bifurcation When $0 \le r \le 4$ **(B) Logistic Bifurcation When** $1 \le r \le 4$.

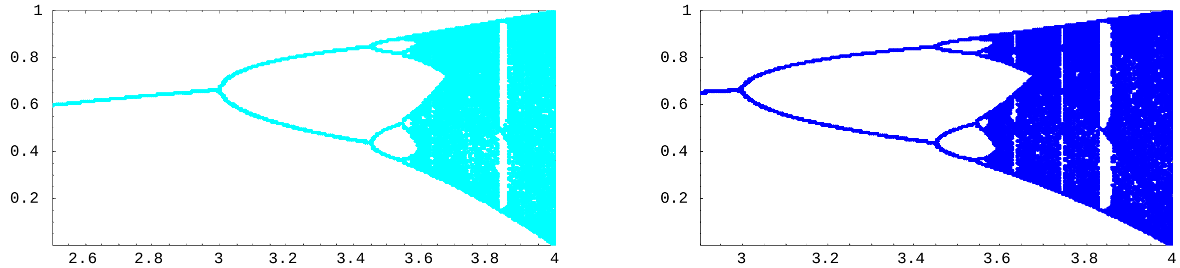

Figure 5: (C) Logistic Bifurcation When $2.5 \le r \le 4$, **(D) Logistic Bifurcation When** $2.90 \le r \le 4$.

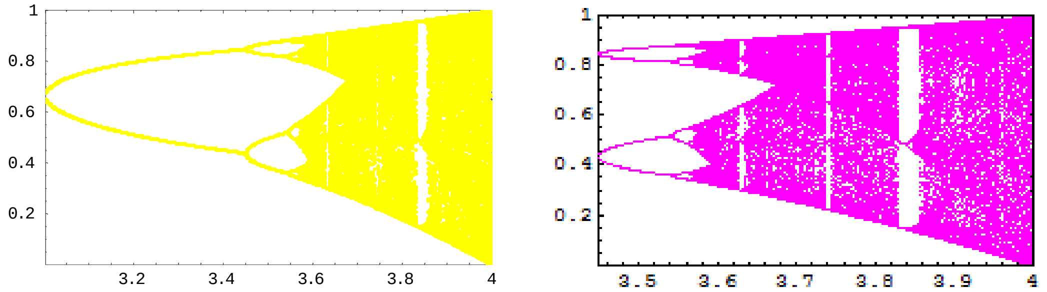

Figure 5: (E) Logistic Bifurcation When $3 \le r \le 4$, **(F) Logistic Bifurcation When** $3 \le r \le 4$.

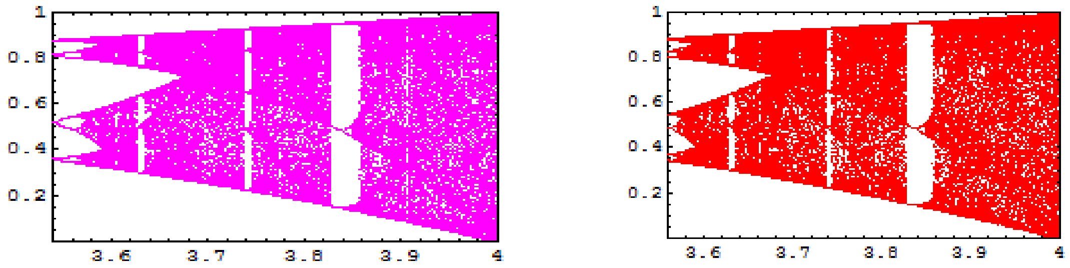

Figure 5: (G) Logistic Bifurcation when $3.54 \le r \le 4$, **(H) Logistic Bifurcation when** $3.56 \le r \le 4$.

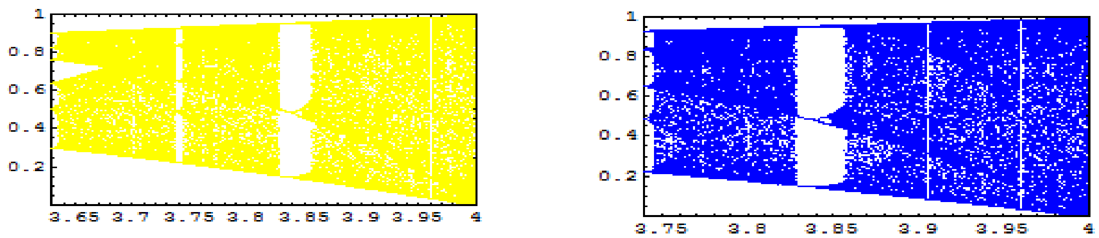

Figure 5: (I) Logistic Bifurcation When $3.63 \le r \le 4$, **(J) Logistic Bifurcation When** $3.74 \le r \le 4$.

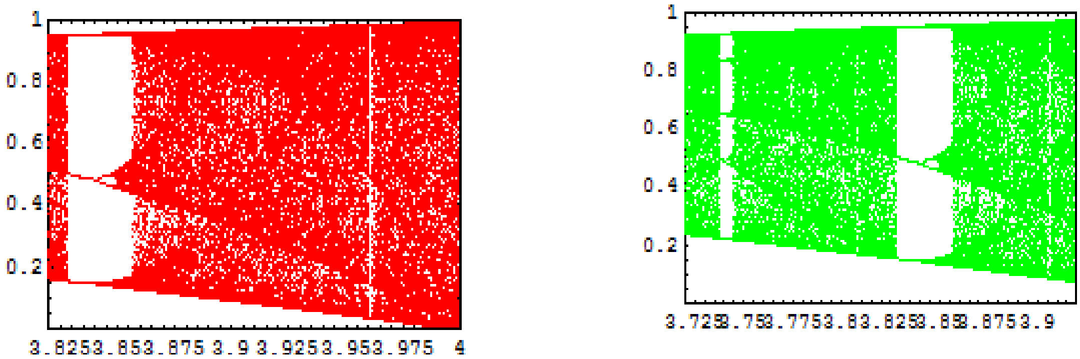

Figure 5: (K) Logistic Bifurcation When $3.82 \leq r \leq 4$**, (L) Logistic Bifurcation When** $3.72 \leq r \leq 3.92$**.**

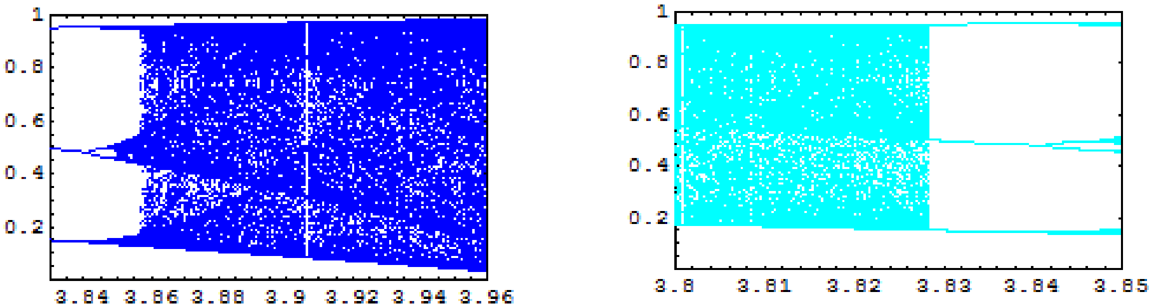

Figure 5 (M) Logistic Bifurcation When $3.83 \leq r \leq 3.96$**, (N) Logistic Bifurcation When** $3.80 \leq r \leq 3.85$**.**

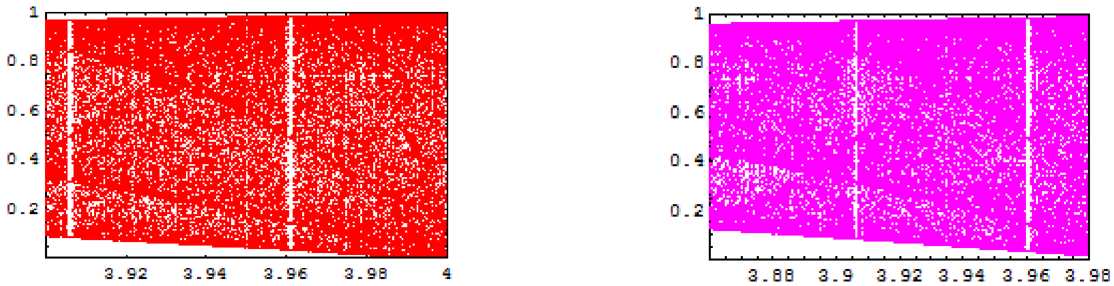

Figure 5: (O) Logistic Bifurcation When $3.90 \leq r \leq 4$**, (P) Logistic Bifurcation When** $3.86 \leq r \leq 3.98$.

CONCLUSIONS

We have viewed sundry initial convictions of bifurcation, saddle-node bifurcation, period doubling bifurcation. Some observations of logistic map, animation of a point "rolling" along the logistic curve, bifurcation diagram on logistic map for several iterations are illustrated. Finally, we detached the bifurcation diagram of logistic map and their chaotic and non-chaotic ambiences for isolated parameter values.

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